Journal of MATHEMATICAL
www.elsevier.com/locate/jmaa

The convolution of functions and distributions

Brian Fisher [a,*], Kenan Taş [b]

[a] *Department of Mathematics, University of Leicester, Leicester, LE1 7RH, England, UK*
[b] *Department of Mathematics, Çankaya University, Ankara, Turkey*

Received 23 June 2004

Available online 29 January 2005

Submitted by B.S. Thomson

Abstract

The non-commutative convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ is defined to be the limit of the sequence $\{(f\tau_n) * g\}$, provided the limit exists, where $\{\tau_n\}$ is a certain sequence of functions in $\mathcal{D}$ converging to 1. It is proved that

$$|x|^\lambda * \left(\operatorname{sgn} x|x|^\mu\right) = \frac{2\sin(\lambda\pi/2)\cos(\mu\pi/2)}{\sin[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1) \operatorname{sgn} x|x|^{\lambda+\mu+1},$$

for $-1 < \lambda + \mu < 0$ and $\lambda, \mu \neq -1, -2, \ldots,$ where B denotes the Beta function.

Keywords: Distribution; Dirac delta function; Convolution

In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see for example Gel'fand and Shilov [1].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:

* Corresponding author.
E-mail addresses: fbr@le.ac.uk (B. Fisher), kenan@cankaya.edu.tr (K. Taş).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.004

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

Then the *convolution* $f * g$ is defined by the equation

$$\big\langle (f*g)(x), \varphi(x)\big\rangle = \big\langle g(x), \big\langle f(t), \varphi(x+t)\big\rangle\big\rangle \tag{1}$$

for arbitrary test function φ in $\mathcal{D}$.

The classical definition of the convolution is as follows:

Definition 2. If f and g are locally summable functions, then the *convolution* $f * g$ is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt \tag{2}$$

for all x for which the integrals exist.

Note that if f and g are locally summable functions satisfying either of the conditions (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2.

It follows that if the convolution $f * g$ exists by Definitions 1 or 2, then the following equations hold:

$$f*g = g*f, \tag{3}$$

$$(f*g)' = f*g' = f'*g. \tag{4}$$

Definition 1 is rather restrictive and so a neutrix convolution was introduced in [2]. In order to define the neutrix convolution, we first of all let τ be the function in $\mathcal{D}$, see Jones [3], satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leqslant \tau(x) \leqslant 1$,
(iii) $\tau(x) = 1$, $|x| \leqslant \frac{1}{2}$,
(iv) $\tau(x) = 0$, $|x| \geqslant 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution* $f \circledast g$ is defined to be the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [4], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero as n tends to infinity.

Note that the convolution $f_n * g$ in this definition is in the sense of Definition 2, the support of f_n being bounded. Note also that the neutrix convolution in this definition, is in general non-commutative.

It was proved in [2] that if the convolution $f * g$ exists by Definition 1, then the neutrix convolution $f \circledast g$ exists and

$$f * g = f \circledast g,$$

showing that Definition 3 is a generalization of Definition 1.

We now give a definition of the convolution which generalizes both Definitions 1 and 2 but is a particular case of Definition 3.

Definition 4. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *convolution* $f * g$ is defined to be the limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$.

From now on, we will use Definition 4 for the definition of the convolution.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the convolution $f * g$ exists. Then the convolution $f * g'$ exists and*

$$(f * g)' = f * g'. \tag{5}$$

Further, if $\lim_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$, then the convolution $f' * g$ exists and*

$$(f * g)' = f' * g + h. \tag{6}$$

Proof. Suppose that $f * g$ exists. Since f_n has compact support, Eq. (4) holds and so

$$\big\langle (f_n * g)', \varphi\big\rangle = \langle f_n * g', \varphi\rangle \tag{7}$$

for all φ in $\mathcal{D}$. Equation (5) follows on letting n tend to infinity in Eq. (7).

Next we have

$$\big\langle (f_n * g)', \varphi\big\rangle = \big\langle (f_n)' * g, \varphi\big\rangle = \big\langle (f')_n * g + (f\tau_n') * g, \varphi\big\rangle \tag{8}$$

for all φ in $\mathcal{D}$. Equation (6) follows on letting n tend to infinity in Eq. (8). $\square$

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the convolution $f' * g$ exists and $\lim_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$. Then the convolution $f * g'$ exists and*

$$f * g' = f' * g + h. \tag{9}$$

Alternatively, if $f * g'$ exists, then the convolution $f' * g$ exists and

$$f' * g = f * g' - h. \tag{10}$$

Proof. Suppose that $f' * g$ exists. Since f_n has compact support, Eq. (4) holds and so

$$\langle f_n * g', \varphi\rangle = \big\langle (f_n)' * g, \varphi\big\rangle = \big\langle (f')_n * g + (f\tau_n') * g, \varphi\big\rangle \tag{11}$$

for all φ in $\mathcal{D}$. Equation (9) follows on letting n tend to infinity in Eq. (11).

If now $f * g'$ exists, then Eq. (10) follows on letting n tend to infinity in Eq. (11). □

We now prove our main theorem.

Theorem 3. *The convolutions $|x|^\lambda * (\operatorname{sgn} x|x|^\mu)$ and $(\operatorname{sgn} x|x|^\lambda) * |x|^\mu$ exist and*

$$|x|^\lambda * \big(\operatorname{sgn} x|x|^\mu\big) = \frac{2\sin(\lambda\pi/2)\cos(\mu\pi/2)}{\sin[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1)\operatorname{sgn} x|x|^{\lambda+\mu+1}, \tag{12}$$

$$\big(\operatorname{sgn} x|x|^\lambda\big) * |x|^\mu = \frac{2\sin(\mu\pi/2)\cos(\lambda\pi/2)}{\sin[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1)\operatorname{sgn} x|x|^{\lambda+\mu+1} \tag{13}$$

for $-1 < \lambda+\mu < 0$ and $\lambda, \mu \neq -1, -2, \ldots$.

Proof. We will first of all suppose that $\lambda, \mu > -1$ with $-1 < \lambda+\mu < 0$, and put

$$|x|_n^\lambda = |x|^\lambda \tau_n(x), \qquad \big(x_+^\lambda\big)_n = x_+^\lambda \tau_n(x), \qquad \big(x_-^\lambda\big)_n = x_-^\lambda \tau_n(x).$$

Then

$$\begin{aligned} |x|_n^\lambda * \big(\operatorname{sgn} x|x|^\mu\big) &= \big[\big(x_+^\lambda\big)_n + \big(x_-^\lambda\big)_n\big] * \big(x_+^\mu - x_-^\mu\big) \\ &= \big(x_+^\lambda\big)_n * x_+^\mu + \big(x_-^\lambda\big)_n * x_+^\mu - \big(x_+^\lambda\big)_n * x_-^\mu - \big(x_-^\lambda\big)_n * x_-^\mu \\ &= I_1 + I_2 - I_3 - I_4, \end{aligned} \tag{14}$$

the convolutions existing by Definition 1. It is clear that

$$\lim_{n\to\infty} I_1 = x_+^\lambda * x_+^\mu = B(\lambda+1, \mu+1)x_+^{\lambda+\mu+1}, \tag{15}$$

$$\lim_{n\to\infty} I_4 = x_-^\lambda * x_-^\mu = B(\lambda+1, \mu+1)x_-^{\lambda+\mu+1}, \tag{16}$$

where B denotes the Beta function. Equations (15) and (16) in fact exist for all $\lambda, \mu > -1$ by Definition 2 and for all $\lambda, \mu, \lambda+\mu+1 \neq -1, -2, \ldots$ by Definition 1.

Further,

$$\big(x_-^\lambda\big)_n * x_+^\mu = \int_{-n}^{0} |t|^\lambda (x-t)_+^\mu \, dt + \int_{-n-n^{-n}}^{-n} |t|^\lambda (x-t)_+^\mu \tau_n(t)\, dt. \tag{17}$$

If $-n < x < 0$, we have on making the substitution $t = xu^{-1}$,

$$\begin{aligned}
\int_{-n}^{0} |t|^{\lambda}(x-t)_+^{\mu}\,dt &= \int_{-n}^{x} |t|^{\lambda}(x-t)^{\mu}\,dt \\
&= |x|^{\lambda+\mu+1} \int_{-x/n}^{1} u^{-\lambda-\mu-2}(1-u)^{\mu}\,du \\
&= |x|^{\lambda+\mu+1} \int_{-x/n}^{1} u^{-\lambda-\mu-2}\big[(1-u)^{\mu}-1\big]\,du \\
&\quad + |x|^{\lambda+\mu+1}\frac{1-|n/x|^{\lambda+\mu+1}}{-\lambda-\mu-1}.
\end{aligned} \tag{18}$$

This equation shows that the convolution $x_-^{\lambda} * x_+^{\mu}$ does not exist if $\lambda+\mu > -1$.

If $x > 0$, we have on making the substitution $t = x(1-u^{-1})$,

$$\begin{aligned}
\int_{-n}^{0} |t|^{\lambda}(x-t)_+^{\mu}\,dt &= \int_{-n}^{0} |t|^{\lambda}(x-t)^{\mu}\,dt \\
&= x^{\lambda+\mu+1} \int_{x/(x+n)}^{1} u^{-\lambda-\mu-2}(1-u)^{\lambda}\,du \\
&= x^{\lambda+\mu+1} \int_{x/(x+n)}^{1} u^{-\lambda-\mu-2}\big[(1-u)^{\lambda}-1\big]\,du \\
&\quad + x^{\lambda+\mu+1}\frac{1-[(x+n)/x]^{\lambda+\mu+1}}{-\lambda-\mu-1}.
\end{aligned} \tag{19}$$

It is easily seen that

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} |t|^{\lambda}(x-t)_+^{\mu}\tau_n(t)\,dt = 0 \tag{20}$$

for all x.

Similarly,

$$\big(x_+^{\lambda}\big)_n * x_-^{\mu} = \int_{0}^{n} t^{\lambda}(x-t)_-^{\mu}\,dt + \int_{n}^{n+n^{-n}} t^{\lambda}(x-t)_-^{\mu}\tau_n(t)\,dt. \tag{21}$$

If $n > x > 0$, we have on making the substitution $t = xu^{-1}$,

$$\int_0^n t^\lambda (x-t)_-^\mu \, dt = \int_x^n t^\lambda (t-x)^\mu \, dt$$
$$= x^{\lambda+\mu+1} \int_{x/n}^1 u^{-\lambda-\mu-2} \big[(1-u)^\mu - 1\big] \, du$$
$$+ x^{\lambda+\mu+1} \frac{1-(n/x)^{\lambda+\mu+1}}{-\lambda-\mu-1}. \tag{22}$$

If $x < 0$, we have on making the substitution $t = x(1-u^{-1})$,

$$\int_0^n t^\lambda (x-t)_-^\mu \, dt = \int_0^n t^\lambda (t-x)^\mu \, dt$$
$$= |x|^{\lambda+\mu+1} \int_{x/(x-n)}^1 u^{-\lambda-\mu-2} (1-u)^\lambda \, du$$
$$= |x|^{\lambda+\mu+1} \int_{x/(x-n)}^1 u^{-\lambda-\mu-2} \big[(1-u)^\lambda - 1\big] \, du$$
$$+ |x|^{\lambda+\mu+1} \frac{1-[(x-n)/x]^{\lambda+\mu+1}}{-\lambda-\mu-1}. \tag{23}$$

It is easily seen that

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} t^\lambda (x-t)_+^\mu \tau_n(t) \, dt = 0 \tag{24}$$

for all x.

It now follows from Eqs. (18) and (23) that if $-n < x < 0$, then

$$\int_{-n}^0 |t|^\lambda (x-t)_+^\mu \, dt - \int_0^n t^\lambda (x-t)_-^\mu \, dt$$
$$= |x|^{\lambda+\mu+1} \int_{-x/n}^1 u^{-\lambda-\mu-2} \big[(1-u)^\mu - 1\big] \, du - \frac{|x|^{\lambda+\mu+1}}{\lambda+\mu+1}$$
$$- |x|^{\lambda+\mu+1} \int_{x/(x-n)}^1 u^{-\lambda-\mu-2} \big[(1-u)^\lambda - 1\big] \, du + \frac{|x|^{\lambda+\mu+1}}{\lambda+\mu+1} + O\big(n^{\lambda+\mu}\big) \tag{25}$$

and so

$$\begin{aligned}\lim_{n\to\infty}(I_2-I_3) &= \lim_{n\to\infty}\left[\int_{-n}^{0}|t|^{\lambda}(x-t)_+^{\mu}\,dt-\int_0^n t^{\lambda}(x-t)_-^{\mu}\,dt\right]\\ &= \big[B(-\lambda-\mu-1,\mu+1)-B(-\lambda-\mu-1,\lambda+1)\big]|x|^{\lambda+\mu+1}, \qquad (26)\end{aligned}$$

on using Eqs. (17), (20), (21) and (24), see Gel'fand and Shilov [1].

Similarly, it follows from Eqs. (17), (19), (21) and (22) that if $n > x > 0$, then

$$\begin{aligned}&\int_{-n}^{0}|t|^{\lambda}(x-t)_+^{\mu}\,dt-\int_0^n t^{\lambda}(x-t)_-^{\mu}\,dt\\ &\quad= x^{\lambda+\mu+1}\int_{x/(x+n)}^{1}u^{-\lambda-\mu-2}\big[(1-u)^{\lambda}-1\big]\,du-\frac{x^{\lambda+\mu+1}}{\lambda+\mu+1}\\ &\qquad- x^{\lambda+\mu+1}\int_{x/n}^{1}u^{-\lambda-\mu-2}\big[(1-u)^{\mu}-1\big]\,du+\frac{x^{\lambda+\mu+1}}{\lambda+\mu+1}+O\big(n^{\lambda+\mu}\big)\end{aligned}$$

and so

$$\begin{aligned}\lim_{n\to\infty}(I_2-I_3) &= \lim_{n\to\infty}\left[\int_{-n}^{0}|t|^{\lambda}(x-t)_+^{\mu}\,dt-\int_0^n t^{\lambda}(x-t)_-^{\mu}\,dt\right]\\ &= \big[B(-\lambda-\mu-1,\lambda+1)-B(-\lambda-\mu-1,\mu+1)\big]x^{\lambda+\mu+1} \qquad (27)\end{aligned}$$

on using Eqs. (17), (20), (21) and (24).

It now follows from Eqs. (14) to (16), (26) and (27) that

$$\begin{aligned}\lim_{n\to\infty}|x|_n^{\lambda}*\big(\operatorname{sgn}x|x|^{\mu}\big) &= |x|^{\lambda}*\big(\operatorname{sgn}x|x|^{\mu}\big)\\ &= \big[B(\lambda+1,\mu+1)+B(-\lambda-\mu-1,\lambda+1)\\ &\quad - B(-\lambda-\mu-1,\mu+1)\big]\operatorname{sgn}x|x|^{\lambda+\mu+1}. \qquad (28)\end{aligned}$$

Now, if $\mu \neq 0$,

$$\begin{aligned}B(-\lambda-\mu-1,\lambda+1) &= \frac{\Gamma(-\lambda-\mu-1)\Gamma(\lambda+1)}{\Gamma(-\mu)}\\ &= -\frac{\Gamma(\lambda+1)\Gamma(\mu+1)\sin(\mu\pi)}{\Gamma(\lambda+\mu+2)\sin[(\lambda+\mu)\pi]}\\ &= -\frac{B(\lambda+1,\mu+1)\sin(\mu\pi)}{\sin[(\lambda+\mu)\pi]}, \qquad (29)\end{aligned}$$

where Γ denotes the Gamma function, and if $\mu = 0$,

$$B(-\lambda-1,\lambda+1)=0,$$

which is in agreement with Eq. (29).

Similarly,

$$B(-\lambda-\mu-1,\lambda+1) = -\frac{B(\lambda+1,\lambda+1)\sin(\lambda\pi)}{\sin[(\lambda+\mu)\pi]},$$

and so

$$\begin{aligned}
&B(\lambda+1,\mu+1)+B(-\lambda-\mu-1,\lambda+1)-B(-\lambda-\mu-1,\mu+1)\\
&\quad=\left[1+\frac{\sin(\lambda\pi)-\sin\mu\pi)}{\sin[(\lambda+\mu)\pi]}\right]B(\lambda+1,\mu+1)\\
&\quad=\frac{2\sin(\lambda\pi/2)\cos(\mu\pi/2)}{\sin[(\lambda+\mu)\pi/2]}B(\lambda+1,\mu+1). \qquad (30)
\end{aligned}$$

Equation (12) now follows from Eqs. (28) and (30) for $-1<\lambda+\mu<0$ and $\lambda,\mu>-1$.

Similarly, putting $(\operatorname{sgn} x|x|^\lambda)_n = (\operatorname{sgn} x|x|^\lambda)\tau_n$, we have

$$\begin{aligned}
\left(\operatorname{sgn} x|x|^\lambda\right)_n * |x|^\mu &= \left(x_+^\lambda\right)_n * x_+^\mu - \left(x_-^\lambda\right)_n * x_+^\mu + \left(x_+^\lambda\right)_n * x_-^\mu - \left(x_-^\lambda\right)_n * x_-^\mu\\
&= I_1 - I_2 + I_3 - I_4
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{n\to\infty}\left(\operatorname{sgn} x|x|^\lambda\right)_n * |x|^\mu &= \left(\operatorname{sgn} x|x|^\lambda\right) * |x|^\mu\\
&= \big[B(\lambda+1,\mu+1)-B(-\lambda-\mu-1,\lambda+1)\\
&\quad + B(-\lambda-\mu-1,\mu+1)\big]\operatorname{sgn} x|x|^{\lambda+\mu+1}\\
&= \frac{2\sin(\mu\pi/2)\cos(\lambda\pi/2)}{\sin[(\lambda+\mu)\pi/2]}B(\lambda+1,\mu+1)\operatorname{sgn} x|x|^{\lambda+\mu+1},
\end{aligned}$$

proving Eq. (13) for $-1<\lambda+\mu<0$ and $\lambda,\mu>-1$.

Now suppose that Eqs. (12) and (13) hold when $-1<\lambda+\mu<0$ and $r-1<\lambda\leqslant r$ for some non-negative integer r. This is true when $r=0$. Then with $|x|<n$, we have

$$\begin{aligned}
\left[|x|^\lambda\tau_n'(x)\right] * \left(\operatorname{sgn} x|x|^\mu\right) &= -\int_n^{n+n^{-n}} t^\lambda(t-x)|^\mu\,d\tau_n(t) + \int_{-n-n^{-n}}^{-n}|t|^\lambda(x-t)^\mu\,d\tau_n(t)\\
&= n^\lambda(n-x)^\mu + n^\lambda(x+n)^\mu\\
&\quad + \int_n^{n+n^{-n}}\left[\lambda t^{\lambda-1}(t-x)^\mu + \mu t^\lambda(t-x)^{\mu-1}\right]\tau_n(t)\,dt\\
&\quad - \int_{-n-n^{-n}}^{-n}\left[-\lambda|t|^{\lambda-1}(x-t)^\mu - \mu t^\lambda(x-t)^{\mu-1}\right]\tau_n(t)\,dt
\end{aligned}$$

and it follows that

$$\lim_{n\to\infty}\left[|x|^\lambda\tau_n'(x)\right] * \left(\operatorname{sgn} x|x|^\mu\right) = 0. \qquad (31)$$

It now follows from Theorem 2, our assumptions and Eq. (31) that

$$\big(|x|^{\lambda+1}\big)' * |x|^{\mu} = |x|^{\lambda+1} * \big(|x|^{\mu}\big)' = \mu|x|^{\lambda+1} * \big(\operatorname{sgn} x|x|^{\mu-1}\big)$$
$$= (\lambda+1)\big(\operatorname{sgn} x|x|^{\lambda}\big) * |x|^{\mu}$$

and so the convolution $|x|^{\lambda+1} * (\operatorname{sgn} x|x|^{\mu-1})$ exists and

$$|x|^{\lambda+1} * \big(\operatorname{sgn} x|x|^{\mu-1}\big) = \frac{\lambda+1}{\mu}\big(\operatorname{sgn} x|x|^{\lambda}\big) * |x|^{\mu}$$
$$= \frac{2(\lambda+1)\sin(\mu\pi/2)\cos(\lambda\pi/2)}{\mu\sin[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1) \operatorname{sgn} x|x|^{\lambda+\mu+1}$$
$$= \frac{2\sin[(\lambda+1)\pi/2]\cos[(\mu-1)\pi/2]}{\sin[(\lambda+\mu)\pi/2]} B(\lambda+2, \mu) \operatorname{sgn} x|x|^{\lambda+\mu+1}.$$

Equation (8) therefore holds for $r < \lambda \leqslant r+1$ and so follows by induction for $\lambda > -1$ and $-1 < \lambda + \mu < 0$.

Similarly, Eq. (13) holds for $\lambda > -1$ and $-1 < \lambda + \mu < 0$.

A similar induction argument proves that Eqs. (12) and (13) hold for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $-1 < \lambda + \mu < 0$.

This completes the proof of the theorem. □

Particular cases of Eqs. (12) and (13) are

$$x^{2r} * \big(\operatorname{sgn} x|x|^{\mu}\big) = \big(\operatorname{sgn} x|x|^{\mu}\big) * x^{2r} = 0$$

for $r = 0, 1, 2, \ldots$ and $-1 < -2r + \mu < 0$ and

$$|x|^{2r+1} * \big(\operatorname{sgn} x|x|^{\mu}\big) = \big(\operatorname{sgn} x|x|^{\mu}\big) * |x|^{2r+1} = 2B(2r+2, \mu+1) \operatorname{sgn} x|x|^{2r+2+\mu}$$

for $r = 0, 1, 2, \ldots$ and $-1 < 2r + 1 + \mu < 0$.

Theorem 4. *The convolutions $x_-^{\lambda} * x_+^{\mu}$ and $x_+^{\lambda} * x_-^{\mu}$ exist and*

$$x_-^{\lambda} * x_+^{\mu} = B(-\lambda-\mu-1, \mu+1)x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1)x_+^{\lambda+\mu+1}, \tag{32}$$
$$x_+^{\lambda} * x_-^{\mu} = B(-\lambda-\mu-1, \mu+1)x_+^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1)x_-^{\lambda+\mu+1} \tag{33}$$

for $\lambda + \mu < -1$ and $\lambda, \mu, \lambda + \mu \neq -1, -2, \ldots$.

Proof. Suppose first of all that $-2 < \lambda + \mu < -1$ and $\lambda, \mu > -1$. Then it follows from Eqs. (18) and (19) that

$$\lim_{n\to\infty} \int_{-n}^{0} |t|^{\lambda}(x-t)_+^{\mu} = B(-\lambda-\mu-1, \mu+1)|x|^{\lambda+\mu+1}, \tag{34}$$

if $x < 0$ and

$$\lim_{n\to\infty} \int_{-n}^{0} |t|^{\lambda}(x-t)_+^{\mu} = B(-\lambda-\mu-1, \mu+1)x^{\lambda+\mu+1}, \tag{35}$$

if $x > 0$, since $\lambda + \mu + 1 < 0$. Equation (32) now follows from Eqs. (17), (20), (34) and (35) for $-2 < \lambda + \mu < -1$ and $\lambda, \mu > -1$.

Equation (33) follows on replacing x by $-x$ in Eq. (32) for $-2 < \lambda + \mu < -1$ and $\lambda, \mu > -1$.

Induction arguments similar to those given above now prove that Eqs. (32) and (33) hold for $-2 < \lambda + \mu < -1$ and $\lambda, \mu \neq -1, -2, \dots$.

Now suppose that Eqs. (32) and (33) hold for $-r - 1 < \lambda + \mu < -r$ and $\lambda, \mu \neq -1, -2, \dots$ for some positive integer r. This is true when $r = 1$. Then with $|x| < n$, we have

$$\begin{aligned}
\left[x_-^\lambda \tau_n'(x)\right] * x_+^\mu &= \int_{-n-n^{-n}}^{-n} |t|^\lambda (x-t)^\mu \, d\tau_n(t) \\
&= n^\lambda (x+n)^\mu - \int_{-n-n^{-n}}^{-n} \left[-\lambda |t|^{\lambda-1}(x-t)^\mu - \mu t^\lambda (x-t)^{\mu-1}\right]\tau_n(t)\, dt
\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} \left[x_-^\lambda \tau_n'(x)\right] * x_+^\mu = 0. \tag{36}$$

It now follows from Theorem 2, our assumptions and Eq. (36) that

$$\begin{aligned}
\left(x_-^\lambda * x_+^\mu\right)' &= -\lambda x_-^{\lambda-1} * x_+^\mu \\
&= -(\lambda+\mu+1)B(-\lambda-\mu-1, \mu+1)x_-^{\lambda+\mu} \\
&\quad + (\lambda+\mu+1)B(-\lambda-\mu-1, \lambda+1)x_+^{\lambda+\mu}
\end{aligned}$$

and so the convolution $x_-^{\lambda-1} * x_+^\mu$ exists and

$$\begin{aligned}
x_-^{\lambda-1} * x_+^\mu &= \frac{\lambda+\mu+1}{\lambda} B(-\lambda-\mu-1, \mu+1)x_-^{\lambda+\mu} \\
&\quad - \frac{\lambda+\mu+1}{\lambda}(\lambda+\mu+1)B(-\lambda-\mu-1, \lambda+1)x_+^{\lambda+\mu} \\
&= B(-\lambda-\mu, \mu+1)x_-^{\lambda+\mu} + B(-\lambda-\mu, \lambda)x_+^{\lambda+\mu}.
\end{aligned}$$

Equation (32) therefore holds for $-r - 2 < \lambda + \mu < -r - 1$ and so follows by induction for $\lambda + \mu < -1$ and $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

Replacing x by $-x$ in Eq. (32) gives Eq. (33). This completes the proof of the theorem. □

Note that it now follows immediately from Eq. (14) that Eqs. (12) and (13) hold for $\lambda + \mu < -1$ and $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

Corollary 4.1. *The convolutions* $|x|^\lambda * |x|^\mu$ *and* $(\operatorname{sgn} x|x|^\lambda) * (\operatorname{sgn} x|x|^\mu)$ *exist and*

$$|x|^\lambda * |x|^\mu = -\frac{2\sin(\lambda\pi/2)\sin(\mu\pi/2)}{\cos[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1)|x|^{\lambda+\mu+1}, \tag{37}$$

$$\left(\operatorname{sgn} x|x|^\lambda\right) * \left(\operatorname{sgn} x|x|^\mu\right) = \frac{2\cos(\lambda\pi/2)\cos(\mu\pi/2)}{\cos[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1)|x|^{\lambda+\mu+1} \tag{38}$$

for $\lambda+\mu < -1$ *and* $\lambda, \mu, \lambda+\mu \neq -1, -2, \ldots$.

Proof. We have

$$\begin{aligned}
|x|^\lambda * |x|^\mu) &= \left(x_+^\lambda + x_-^\lambda\right) * \left(x_+^\mu + x_-^\mu\right) \\
&= x_+^\lambda * x_+^\mu + x_+^\lambda * x_-^\mu + x_-^\lambda * x_+^\mu + x_-^\lambda * x_-^\mu \\
&= \big[B(\lambda+1, \mu+1) + B(-\lambda-\mu-1, \mu+1) \\
&\quad + B(-\lambda-\mu-1, \lambda+1)\big]|x|^{\lambda+\mu+1} \\
&= -\frac{2\sin(\lambda\pi/2)\sin(\mu\pi/2)}{\cos[(\lambda+\mu)\pi/2]} B(\lambda+1, \mu+1)|x1^{\lambda+\mu+1},
\end{aligned}$$

proving Eq. (37).

Equation (38) follows similarly. □

References

[1] I.M. Gel'fand, G.E. Shilov, Generalized Functions, vol. I, Academic Press, San Diego, 1964.

[2] B. Fisher, Neutrices and the convolution of distributions, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. Novi Sad 17 (1987) 119–135.

[3] D.S. Jones, The convolution of generalized functions, Quart. J. Math. Oxford Ser. (2) 24 (1973) 145–163.

[4] J.G. van der Corput, Introduction to the neutrix calculus, J. Anal. Math. 7 (1959–1960) 291–398.